# Extended Essay – Mathematics

# Alhazen's Billiard Problem

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Word Count: 3017

**Abstract**

The research question of this Mathematics Extended Essay is, "on a circular table there are two balls; at what point along the circumference must one be aimed at in order for it to strike the other after rebounding off the edge". In investigating this question, I first used my own initial approach (which involved measuring various chord lengths), followed by looking at a number of special cases scenarios (for example when both balls are on the diameter, or equidistant from the center) and finally forming a general solution based on coordinate geometry and trigonometric principles. The investigation included using an idea provided by Heinrich Dorrie and with the use of diagrams and a lengthy mathematical analysis with a large emphasis on trigonometric identities, a solution was found. The conclusion reached is, "if we are given the coordinate plane positions of billiard ball A with coordinates $(x_A, y_A)$ and billiard ball B with coordinates $(x_B, y_B)$, and also the radius of the circle, the solution points are at any of the points of intersection of the circular table with the hyperbola, $(x^2 - y^2)P + (r^2)(yp - xm) + (xy2M)$", where $P = (y_A \cdot x_B + y_B \cdot x_A)$, $M = (y_A \cdot y_B - x_A \cdot x_B)$, $p = (x_A + x_B)$, $m = (y_A + y_B)$ and $r$ is the radius. The solution was verified by considering specific examples through technology such as Autograph software and a TI-84 graphing calculator. Finally I briefly looked at various other solutions to the problem and also considered further research questions.

**Word Count : 234**

## Table of Contents



Extended Essay – Mathematics

Extended Essay – Mathematics
Alhazen's Billiard Problem

**Introduction:**

Regarded as one of the classic problems from two dimensional geometry, Alhazen's Billiard Problem has a truly rich history. The problem is believed to have been first introduced by Greek astronomer Ptolemy back in 150 AD[1] and then eventually noticed by 17th century Arabic mathematician Abu Ali al Hassan ibn Alhaitham (whose name was later Latinized into Alhazen)[2].

Alhazen made reference to this problem in one of his published works entitled *Optics* and presented it in the form, "Find the given point on a spherical mirror at which a ray of light coming from a given point must strike in order to be reflected" [3]. Nowadays, this problem is often referred to as the "Billiard Problem" because it involves locating the point on the edge of a circular billiard table at which a cue ball at a given point must be aimed in order to carom (bounce) once off the edge and strike another ball at a second given point.[4]

The focus question of this extended essay will be:

**On a circular billiards table there are two balls; at what point along the circumference must one be aimed at in order for it to strike the other after rebounding off the edge?**

Heinrich Dörrie also described the problem as "find in a given circle an isosceles

[1] Jack Klaff, "The World May be Divided into Two Types of People – Alhazen's Billiard Problem." Viewed 19 February 2005. <http://www.jackflaff.com/hos.htm>

[2] Heinrich Dörrie, 100 Great Problems of Elementary Mathematics: Their History and Solutions. Dover Publications New York, 1965. 197-200

[3] Dörrie 127.

[4] Eric W Weisstein, "Alhazen's Billiard Problem". Mathworld. Dated 1999. Viewed February 25 2006. <http://mathworld.wolfram.com/AlhazensBilliardProblem.html>

triangle whose legs pass through two given points inside the circle".[5] My primary reason for choosing to investigate this focus question is that the I.B Higher Level Mathematics Programme at our school is at times limited with regards to the study of geometry and trigonometry. Investigating this problem gave me an opportunity to fill this void. That being said, the problem was in itself also very appealing to me as I personally enjoy playing billiards or pool and was eager to find out about the mathematics of the game.

The problem appeared in the Daily Telegraph news in 1997 when Dr Peter Nueman, an Oxford don of Queen's Collage, managed to provide a new solution to the problem. Inspired by early mathematician Descartes, Nueman cleverly translated the billiards table geometry simply into $x$ and $y$ coordinates on two axes.[6] This is a method I intend to use further into my extended essay. Please note that this essay (and the solution to the focus question) is narrowed down to emphasize the algebraic solution to Alhazen's Problem - however in the conclusion, other methods are briefly discussed.

**Pre-examination of the problem:**

The great difficulty with this investigation lies within two concepts. First of all, the balls in question are randomly scattered on the table with no specific locations – in other words our solution would need to be generalized for any set of billiard balls. Second of all, the balls need to be treated as fixed points. To begin this investigation one should first consider where and how many possibilities there can be on a circular pool table that would allow for a ball to strike once off the edge and then hit another ball. Moreover, what exactly characterizes the direction of a ball bouncing off a circular table

[5] Dörrie 127

[6] Highfield, Roger. "Don Solves the Last Puzzle Left by Ancient Greeks." Daily Telegraph. April 1, 1997, Issue 676.



border? The law of reflection states that that the angle of reflection and angle of incidence are equal, with each angle being measured from the normal to the boundary (line indicating the border)[7]. In figure 1, the incident path $\theta_i$ must have an angle equal to the reflected path $\theta_r$.

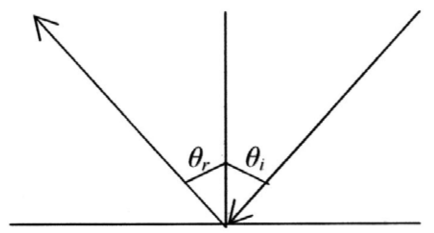
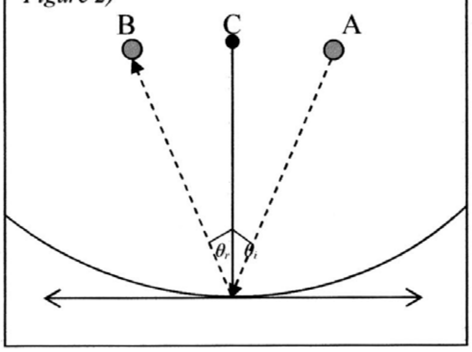



*Figure 1) Law of Reflection,* $\boldsymbol{\theta_i = \theta_r}$

The boundary in our case would be a tangent line drawn to the point on the border of the circle where the ball A bounces off the circular side to ball B (Figure 2).

*Figure 2)*




Another way to express this problem is, "to describe in a given circle an isosceles triangle whose legs pass through two given points made inside the circle"[8]. This is useful because it allows us to relate the billiard balls to chords within the circle. Observe Figure

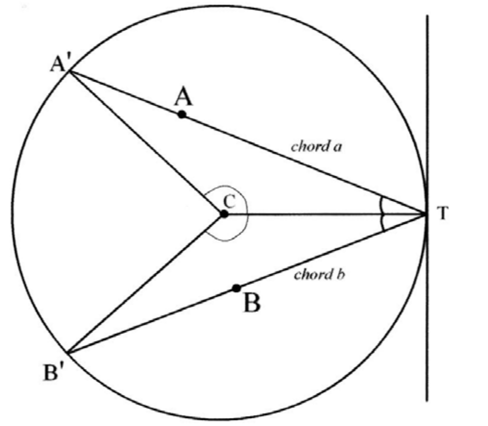
[7] Henderson, Tom. "Reflection and Its Importance". The Physics Classroom. Dated 2004. Viewed 12 March 2005. <http://www.glenbrook.k12.il.us/gbssci/phys/Class/refln/u13l1c.html>

[8] Heinrich Dörrie, 100 Great Problems of Elementary Mathematics: Their History and Solutions. Dover Publications New York, 1965. 197-200

- 5 -

3: ball A and ball B are located within the circular billiard table with the table's center at point C. Ball A needs to make contact with the border at point T in order to strike ball B. If we extend the path that ball A must take to the opposite side of the circle, we have a chord – the same can be done for ball B. The points A' and B' are the second points of intersection of the circle with the respective chords.

*Figure 3)*




If two radii are drawn to the centre of the circle from the points A' and B', we have essentially two triangles – CTA' and CTB'. The length of *chord a* is equal to that of *chord b* for the following reason:

Claim: $A'T \cong B'T$

Proof: 1) $\overline{CT} \cong \overline{CA'} \cong \overline{CB'}$ (radii to circle)

2) $\Delta CTA'$ and $\Delta CTB'$ are isosceles

3) $C\hat{T}A' \cong \Delta C\hat{A}'T$ and $C\hat{T}B' \cong C\hat{B}'T$

4) $C\hat{T}A' \cong C\hat{T}B'$ (angle of incidence = angle of reflection)

5) $A'\hat{C}T \cong B'\hat{C}T$

6) $\Delta A'\hat{C}T \cong \Delta B'\hat{C}T$ (Side-angle-side property)

7) $\therefore A'T \cong B'T$ (corresponding parts of congruent triangles are congruent)

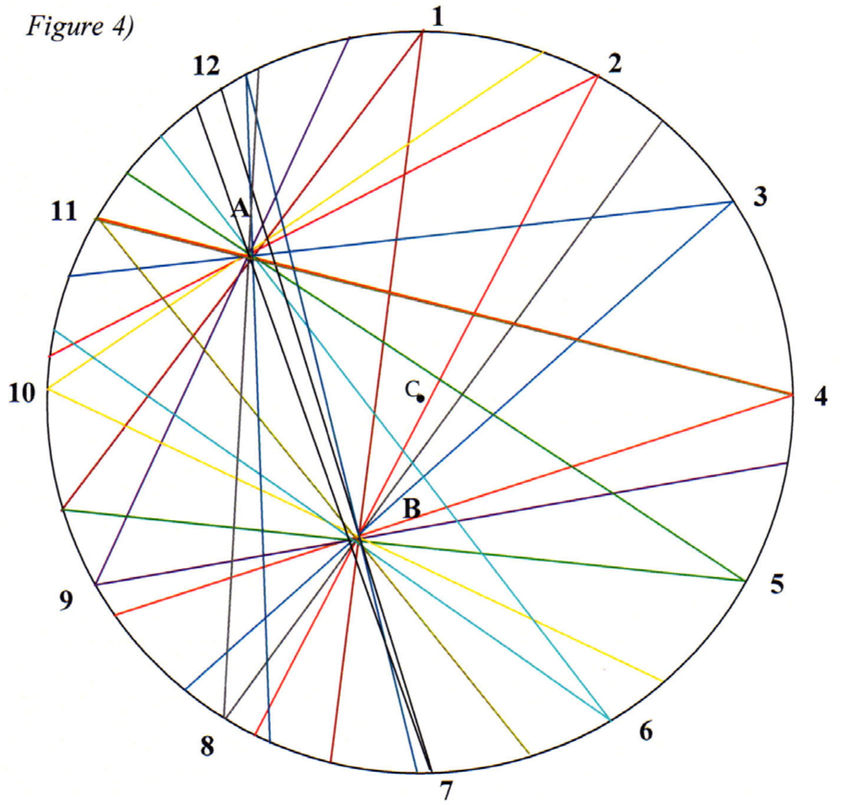
- 6 -

This explains why instead of looking at how one ball must be struck in order for it to strike the other after rebounding off the edge, we can look for an inscribed isosceles triangle whose legs pass through ball A and ball B.

**Initial approach:**

Now in order for us to have a rough idea of the range of possible solutions, we must consider several general cases and see what results we get. Consider Figure 4, here I have randomly chosen two points to be my locations for ball A and ball B. Then I divided the circle into 12 equal parts around the circumference – although the more you divide the circle the more accurate your findings will likely be.

*Figure 4)*



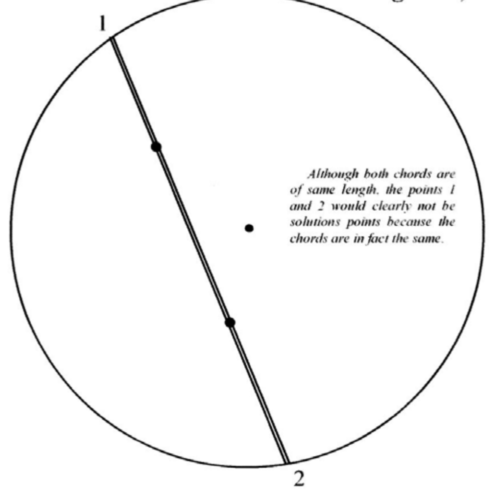

- 7 -

Chords are drawn going through ball A to each of the 12 points, and the same for ball B. The lengths of the chords are measured and recorded in a table (below). The solutions to where ball A must hit to bounce off and hit ball B can possibly be found by looking at where the corresponding chords are equal to one another, in other words where *chord a – chord b = 0*. By making a table showing *chord a – chord b* we could perhaps find possible solutions.

| | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *chord a (cm)* | 9.9 | 10.1 | 11 | 11.8 | 12.3 | 12.1 | 11.6 | 10.6 | 10 | 9.8 | 11.8 | 10.7 |
| *chord b (cm)* | 12.2 | 12.4 | 12.2 | 11.8 | 11.4 | 11.3 | 11.8 | 12.2 | 11.5 | 11.2 | 11.4 | 11.8 |
| *chord a - chord b (cm)* | **-2.3** | **-2.3** | **-2.2** | **0** | **0.9** | **0.8** | **-0.2** | **-1.6** | **-1.5** | **-1.4** | **0.4** | **-1.1** |
| *Solution* | | | | * | | | * | | | * | * | |

From looking at the changing "*chord a – chord b*" we can see that solutions should be at point 4, next to point 7, between points 10 and 11, and between points 11 and 12. However there appears to be an apparent paradox as although our results suggest that there is a solution between the points 11 and 12 and also between the points 6 and 7 on the circumference, by looking at the graph one can see that these chords leading to the points are in fact the same chords and the points would therefore definitely not work as solutions (unless these chords are in fact the diameter, as we will see in the following example).

*Figure 5)*

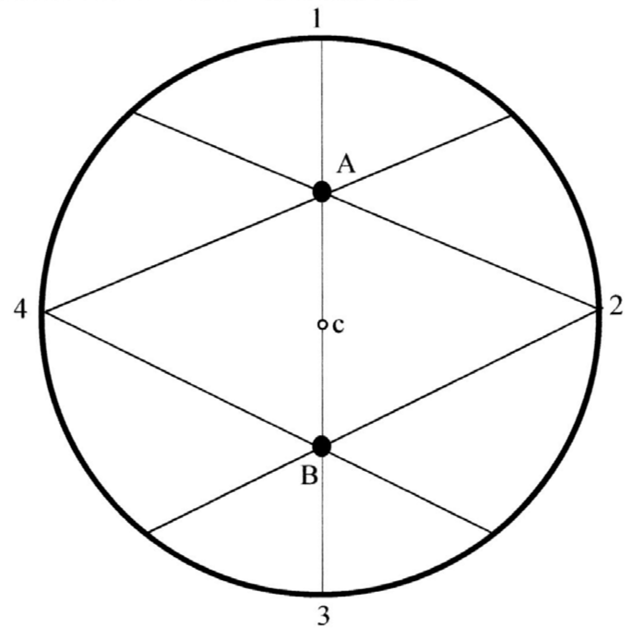



- 8 -

**Analysis of specific scenarios:**

Let us analyze another more specific scenario. In figure 6, ball A and ball B both lie on the diameter, and are equidistant from the centre of the circle at point C.

*Figure 6)*

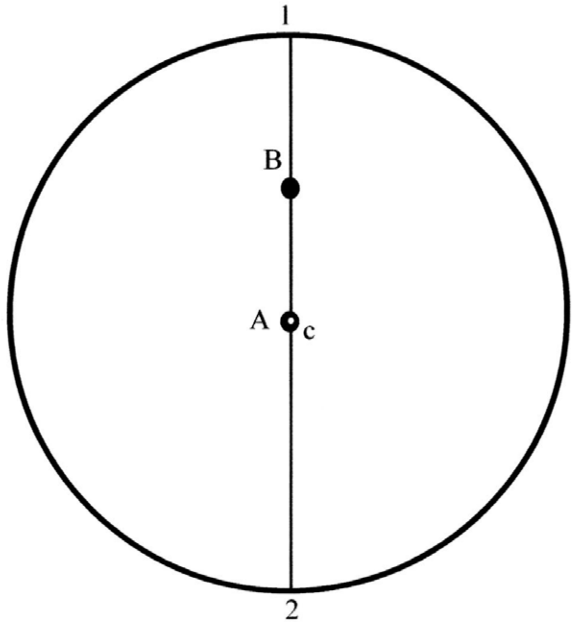



Possible solutions can be found at points 1, 2, 3 and 4 as shown on the diagram. In this case we have four places where we can strike one of the balls so that it rebounds and hits the other. However that being said, if ball A was aimed at point 3, then ball A would "go through" ball B and then bounce off the border to hit the ball back. In other words for ball A, mathematically points 1, 2, 3 and 4 are all solutions, but realistically only points 1, 2, and 4 are solutions because ball B would block the path of ball A before it can reach point 3.

There is another scenario where we arguably only have 2 solutions. In Figure 7 ball A is located exactly at the centre of the circle and ball B is located along the diameter.

- 9 -

*Figure 7)*




If we were striking ball A, the place where it would rebound to then hit ball B is located at point 2. However that being said, if ball A was aimed at point 1, then point 1 would also be a solution. In other words, mathematically speaking both point 1 and point 2 are solutions, but realistically only point 2 is a solution because ball B would block the path of ball A before it can reach point 1.

**Forming a general solution:**

To form a general solution to the problem, I have used an idea provided by Heinrich Dörrie in "100 Great Problems of Elementary Mathematics: Their History and Solutions"[9]. This method is based on using coordinate geometry to form a general solution. I know that the solution points must be on the circumference of the circle (they satisfy the equation of the circle). With this in mind I will attempt to find another equation which also includes the points, using ideas from geometry and trigonometry.

[9] Dörrie 197-200



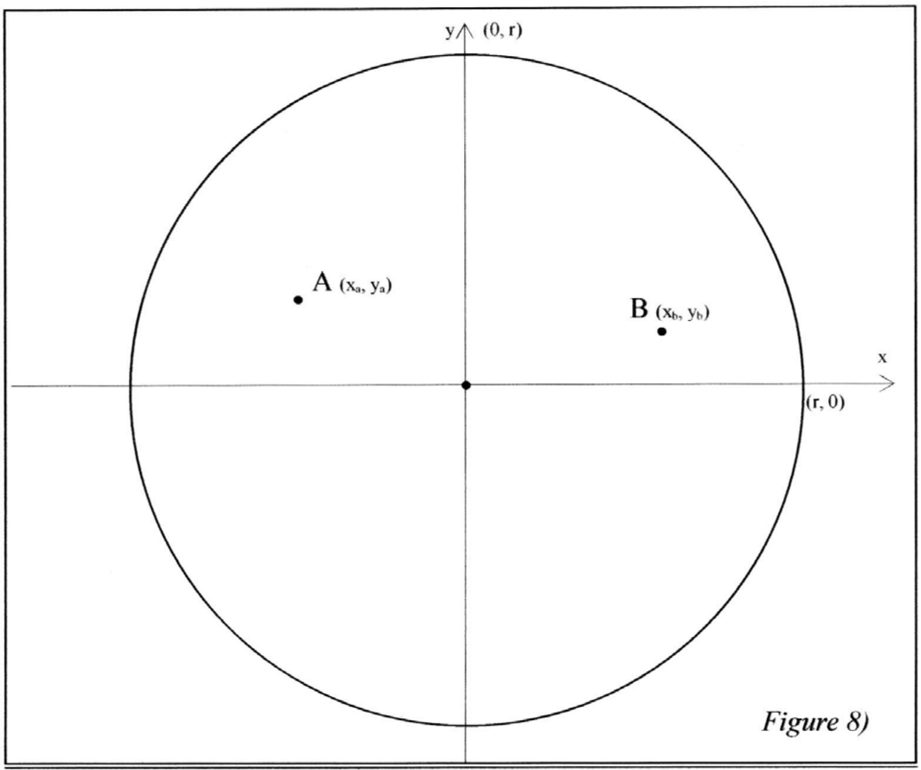



*Figure 8)*

Consider a given circular billiard table with the tables centre at point C and its radius r. Ball A and ball B are randomly scattered on the table. If we then translate this onto a perpendicular coordinate system and make C the origin, then ball A will have the coordinates $(X_a, Y_a)$ and ball B will have the coordinates $(X_b, Y_b)$ as can be seen on Figure 8.

From here we can form an isosceles triangle that includes ball A and ball B (Figure 9 on the next page). Point O is where one of the balls must be aimed at in order to bounce off and hit the other.

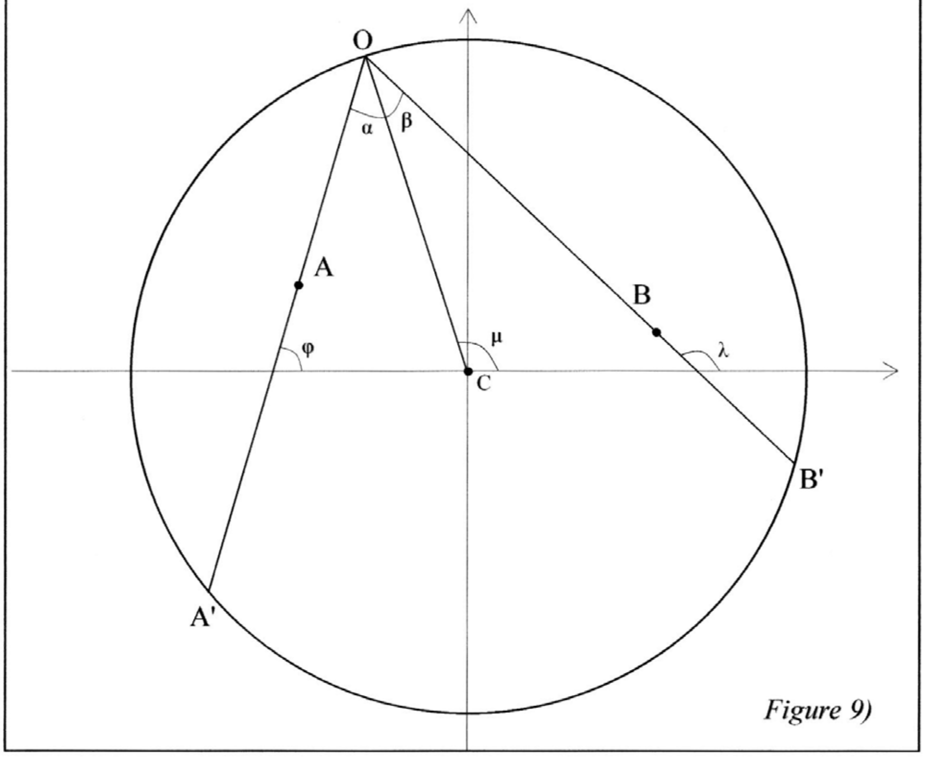
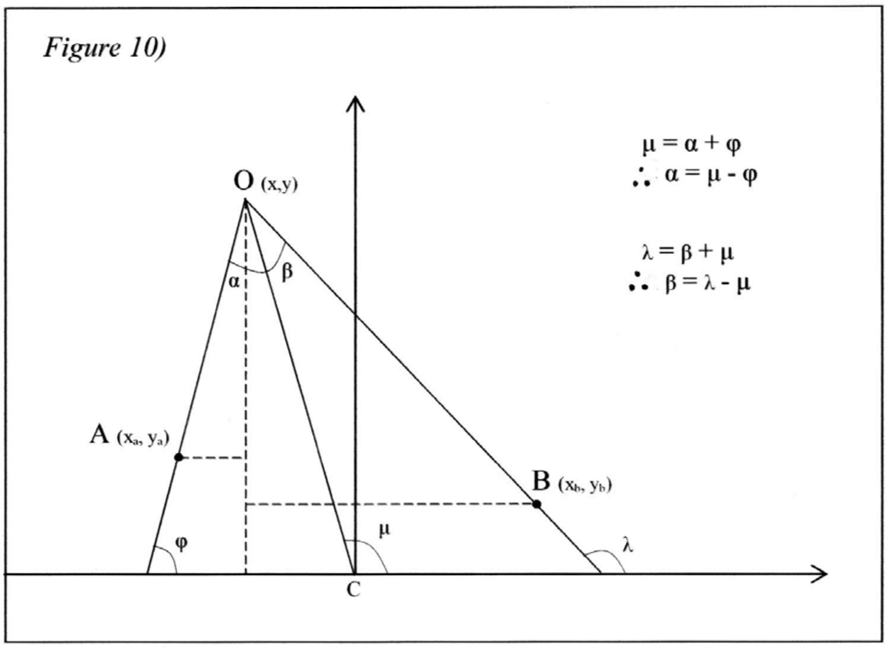



*Figure 9)*

Extending the line OA and OB, we obtain the chords OA' and OB'. Consider the angles that these legs form with the radius OC, angles $\alpha$ and $\beta$ – if ball A is to rebound and strike ball B then $\alpha$ must equal $\beta$ (angle of incidence = angle of reflection). Furthermore, if we extend lines OA, OC, and OB to the circle at A' and B' then we form 3 new angles. The angle between OA' and the x-axis is noted $\varphi$, the angle between OC and the x-axis is $\mu$, and the angle between OB' and the x-axis is $\lambda$.




In Figure 10, it is apparent that $\alpha = \mu - \varphi$ and that $\beta = \lambda - \mu$. Also, from the diagram we can derive that:

$$\tan(\varphi) = \frac{y - y_a}{x - x_a} \qquad \tan(\mu) = \frac{y}{x} \qquad \tan(\gamma) = \frac{y - y_b}{x - x_b}$$

Since angle $\alpha = \beta$

Then $\tan(\alpha) = \tan(\beta)$

We know that $\alpha = \mu - \varphi$ and $\beta = \lambda - \mu$

$$\therefore \tan(\mu - \varphi) = \tan(\lambda - \mu)$$

Using the compound angle identities,

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

We arrive at, $\tan(\mu - \varphi) = \tan(\lambda - \mu)$

$$\frac{\tan(\mu) - \tan(\varphi)}{1 + \tan(\mu)\tan(\varphi)} = \frac{\tan(\lambda) - \tan(\mu)}{1 + \tan(\lambda)\tan(\mu)}$$

- 13 -

now since:

$$\tan(\varphi) = \frac{y - y_a}{x - x_a} \qquad \tan(\mu) = \frac{y}{x} \qquad \tan(\lambda) = \frac{y - y_b}{x - x_b}$$

we have:

$$\frac{\left(\frac{y}{x}\right) - \left(\frac{y - y_A}{x - x_A}\right)}{1 + \left(\frac{y}{x}\right)\left(\frac{y - y_A}{x - x_A}\right)} = \frac{\left(\frac{y - y_B}{x - x_B}\right) - \left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)\left(\frac{y - y_B}{x - x_B}\right)}$$

or,

$$\frac{\frac{y(x - x_A) - x(y - y_A)}{x(x - x_A)}}{\frac{x(x - x_A) + y(y - y_A)}{x(x - x_A)}} = \frac{\frac{x(y - y_B) - y(x - x_B)}{x(x - x_B)}}{\frac{x(x - x_B) + y(y - y_B)}{x(x - x_B)}}$$

or,

$$\frac{yx - yx_A - xy + xy_A}{x^2 - x \cdot x_A + y^2 - y \cdot y_A} = \frac{xy - xy_B - xy + yx_B}{x^2 - x \cdot x_B + y^2 - y \cdot y_B}$$

or,

$$\frac{xy_A - yx_A}{x^2 + y^2 - x \cdot x_A - y \cdot y_A} = \frac{yx_B - xy_B}{x^2 + y^2 - x \cdot x_B - y \cdot y_B}$$

Simplifying,

LHS: $x^2 xy_A + y^2 xy_A - x^2 y_A \cdot x_B - yxy_A \cdot y_B - x^2 yx_A - y^2 yx_A + yx \cdot x_A \cdot x_B + y^2 x_A \cdot y_B$

$=$

RHS: $x^2 yx_B + y^2 yx_B - xyx_A \cdot x_B - y^2 y_A \cdot x_B - x^2 xy_B - xy^2 \cdot y_B + x^2 \cdot x_A \cdot y_B + xy \cdot y_A \cdot y_B$

Bringing everything to one side,

$$- x^2 \cdot xy_A - y^2 xy_A + x^2 y_A \cdot x_B + yx \cdot y_A \cdot y_B + x^2 yx_A + y^2 yx_A - yx \cdot x_A \cdot x_B - y^2 x_A \cdot y_B + x^2 yx_B + y^2 yx_B - xy \cdot x_A \cdot x_B - y^2 y_A \cdot x_B - x^2 xy_B - y^2 xy_B + x^2 x_A \cdot y_B + xy \cdot y_A \cdot y_B = 0$$

- 14 -

or,

$$x^2\left(y_A \cdot x_B + y_B x_A\right) - y^2\left(y_A \cdot x_B + y_B \cdot x_A\right) + y^2\left(yx_A + yx_B - xy_A - xy_B\right)$$
$$+ x^2\left(yx_A + yx_B - xy_A - xy_B\right) + yx\left(y_A \cdot y_B - x_A \cdot x_B - x_A \cdot x_B + y_A \cdot y_B\right) = 0$$

Factoring out,

$$\left(x^2 - y^2\right)\left(y_A \cdot x_B + y_B \cdot x_A\right) + \left(y^2 + x^2\right)\left(yx_A + yx_B - xy_A - xy_B\right) + 2xy\,(y_A \cdot y_B - x_A \cdot x_B) = 0$$

If we assign the following variables:

$$\mathrm{P} = \left(y_A \cdot x_B + y_B \cdot x_A\right), \qquad \mathrm{M} = (y_A \cdot y_B - x_A \cdot x_B)$$

Then we have

$$\left(x^2 - y^2\right)P + \left(y^2 + x^2\right)\left(yx_A + yx_B - xy_A - xy_B\right) + \left(2xyM\right) = 0$$

Further expanding the second bracket of the second sum,

$$\left(x^2 - y^2\right)P + \left(y^2 + x^2\right)\left(y\left(x_A + x_B\right) - x\left(y_A + y_B\right)\right) + \left(2xyM\right) = 0$$

Assigning the following variables:

$$\mathrm{p} = \left(x_A + x_B\right), \qquad \mathrm{m} = \left(y_A + y_B\right)$$

We have the following:

$$\left(x^2 - y^2\right)P + \left(y^2 + x^2\right)(yp - xm) + \left(2xyM\right) = 0$$

**Solution:**

Because the point O(x,y) - where one of the balls must be aimed at in order to bounce off and hit the other - lies on the circle, we can apply the circle equation:

$$x^2 + y^2 = r^2$$

to our equation. Therefore our solution has the form of:

$$\left(x^2 - y^2\right)P + \left(r^2\right)(yp - xm) + \left(2xyM\right) = 0$$

This equation represents a hyperbola (the standard Cartesian hyperbola equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$), therefore our solution is:

- 15 -

*Given the position of both of the billiard balls in a circle and the radius of the table, one can find the solution points at any of the points of intersection of the circle with the hyperbola,* $\left(x^2 - y^2\right) P + \left(r^2\right)(yp - xm) + \left(xy2M\right)$

There are at most four places where a hyperbola intersects with a circle, therefore in general there should be four places where one could strike a ball so that it rebounds and hits the other. However given the nature of a hyperbola, intersections with a circle can occur at only one, two, or even three points (if one of the disconnected curves (arms) of the hyperbola is tangent to the circle) – meaning there will not always be four valid solutions on the circular billiard table.

Relating back to our focus question - if we had two balls in a circular billiard table, and we were able to relate their positions into *(x,y)* coordinates on a plane and also measure the radius of the table, then we would be able to find the exact point(s) along the circumference where one ball must be aimed in order for it to strike the other after rebounding off the edge.

**Verification of Solution**

Now to verify our solution, I will chose a circular table with a radius of 4 units, and two balls with somewhat simple coordinates; ball A with the coordinates (0,1) and ball B with the coordinates (0, -1). The variables we have identified are as follows:

$X_a = 0$, $Y_a = 1$, $X_b = 0$, $Y_b = -1$, $r = 2$

Plugging these into the hyperbola equation we have found, and also into the equation of the circle:

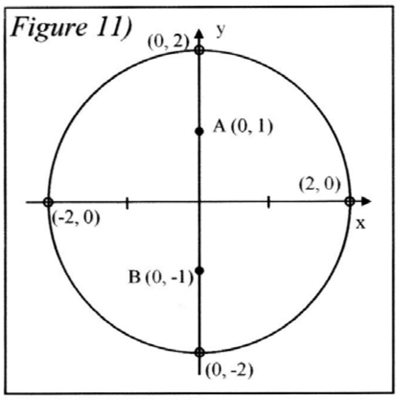
$$(x^2 - y^2)(0+0) + (2^2)(y(0) - x(0)) + (2)(-1)xy) = 0$$

$$\therefore -2xy = 0$$

$\therefore$ *the equation is zero when y = 0, and when x = 0*

- 16 -

Now we will plug these into the equation of the circle: $x^2 + y^2 = 4$




Substituting x = 0

$y^2 = 4$

$y = \pm 2$

Substituting y = 0

$x^2 = 4$

$x = \pm 2$

There are four points along the circumference where one could strike either ball A or ball B and have it rebound to then strike the other ball. These solution points are:

(0, -2), (0, 2), (2, 0), (-2, 0) – as shown in figure 11

If we are to choose points that are slightly more complicated, such as ball A at (-1, 1.5) and ball B at (0.5, 1), then the equation becomes more interesting:

$X_a = -1$, $Y_a = 1.5$, $X_b = 0.5$, $Y_b = 1$, $r = 2$

Plugging all the variables into our found hyperbolic equation, we obtain:

$\{ (x^2 - y^2)((1.5)(0.5) + (1)(-1)) + 4(y(-1 + 0.5) - x(1.5 + 1)) + (2((1.5)(1) - (-1)(0.5))xy) = 0 \}$

$0.25y^2 - 0.25x^2 - 2y - 10x + 4xy = 0$

The equation of the circle is:

$x^2 + y^2 = 4$

Using the equation of the circle:

$y = \pm\sqrt{4 - x^2}$

Now plugging this into the hyperbolic equation (since we are looking for the intersections between the circle and the hyperbola), we obtain two equations:

- 17 -

Using positive square root: $0.25(4-x^2)-0.25x^2-2\sqrt{4-x^2}-10x+4x\sqrt{4-x^2}=0$

Using negative square root: $0.25(4-x^2)-0.25x^2+2\sqrt{4-x^2}-10x-4x\sqrt{4-x^2}=0$

Looking at these equations, it becomes evident that we could find possible solutions using a simple graphing calculator such as the TI-84. The zeros of the functions should give us the x-values of the solution.

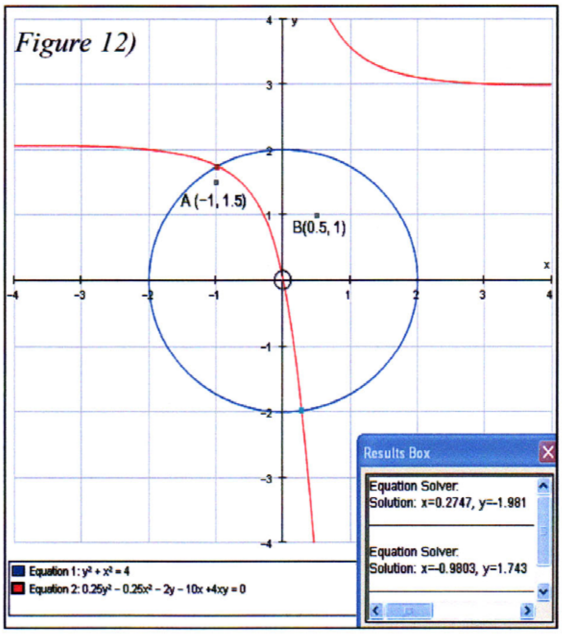
Solution 1 at x = 0.2747 to 4 significant figures

and $y=-\sqrt{4-(0.2747)^2}=-1.981$(4s.f), therefore solution 1 at (0.2747, -1.981)

Solution 2 at x = -0.9803 to 4 decimal places

and $y=\sqrt{4-(-0.9803)^2}=1.743$(4s.f), therefore solution 1 at (-0.9803, 1.743)

Using computer graphing software Autograph 3, we can graph both equations and see how the intersections occur and also find the points of intersection:




Indeed both methods give the same result, and using Autograph 3 we can clearly see how the hyperbola intersects the circle in only two places.

- 18 -

**Other possible solutions:**

This analytical solution just described is only one of many methods known. It answers our focus question and works for any randomly located and infinitely small billiard balls. When plugging in all the known values (such as the radius *r* and the ball locations – *P, p, M* and *m*) one will be able to solve the equation and arrive at one, two, three or four solutions. This will give the point along the circumference must one ball must be aimed at in order for it to strike the other after rebounding off the edge.

Another method introduced by Michael Drexler and Martin J. Gander in their essay "Circular Billiards" involves using a different geometric derivation[10]. They emphasize the fact that for any given ellipse, if a ball is placed in each focus then any point on the rim would be a solution (such is the nature of the foci points). Therefore to solve Alhazen's problem one would need to derive an ellipse that touches the circle (tangent to the circle) and has the two given billiard balls as focal points.

Alhazen's Problem has also been solved using a trisection, as done by Roger C. Alperin in his paper Trisections and Totally Real Origami. Alperin argues that "Alhazen's problem can be solved by Euclidean tools and trisections or equivalently using Origami constructions" [11]. Moreover, modern mathematicians familiar with mathematical computer programs and dynamic geometry software have formulated simple computations that allow one to find solutions to Alhazen's Problem. Luiz Carlos Guimarães and Franck Bellemain argue in their paper entitled "Reflections on the Problema Alhazen" that dynamic geometry software gives an extra touch to geometric

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[10] M Drexler and M Gander. Circular Billiard. SIREV, 1998 volume 40 issue 2. 315 – 323. SIREV Journal 1998.

[11] Alperin, Roger C. Trisections and Totally Real Origami. MAA Monthly 2005. Viewed August 05 2006. < http://www.math.sjsu.edu/%7Ealperin/TRFin.pdf>

 

manipulations, especially with regards to Alhazen's Problem[12]. Although I myself am not greatly familiar with computer geometry software, I can imagine how one could use it with regards to Alhazen's problem. For instance, if a program was instructed to search for same length chords that pass through two given points within a given sized circle, then it would run through all the possibilities, measure them, and find the correct ones (similar to the method I used myself by hand in the early stages of this essay).

Interestingly enough, an easy to use application addressing Alhazen's Problem has been created and can be found freely on the internet. Created by A.I Sabra, the Java application allows one to place two colored dots in any given situation within a spherical concave mirror, and find the maximum amount of points of reflection[13].

**Further Investigation:**

Although Alhazen's Problem might have already been solved and analyzed in a numerous amount of creative ways, one cannot help but wonder how many extended study questions can be proposed relating to Alhazen's Problem. For instance, what would happen if the table were triangular, or hexagonal? Perhaps one could try to interpret Alhazen's Problem not as two points on a two-dimensional circle, but as two points in a three-dimensional sphere.

[12] L. Guimarães ; F. Bellemain. . Reflections on the Problema Alhazeni. The 10th International Congress on Mathematical Education, 2004. Viewed Aug 02 2006. <<www.descartes.ajusco.upn.mx/varios/tsg10/articulos/Bellemain_49_revised_paper.doc>

[13] Sabra, Abdelhamid I. 4 Points of Reflection Applet, "Alhazen's Applet". Viewed August 20 2006. Harvard University. <http://www.people.fas.harvard.edu/~sabra/applets/>

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- 21 -

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