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# Mathematics: analysis and approaches

## Higher level

### Paper 3

6 May 2024

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

**If two functions  $f(x)$  and  $g(x)$  are differentiable, then their product is differentiable and the two functions satisfy the product rule:  $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$ .**

**In this question, you will meet examples of pairs of differentiable functions,  $f(x)$  and  $g(x)$ , that also satisfy  $(f(x)g(x))' = f'(x)g'(x)$ .**

In part (a), consider  $f(x) = \frac{1}{(2-x)^2}$ , where  $x \in \mathbb{R}$ ,  $x \neq 2$ , and  $g(x) = x^2$ , where  $x \in \mathbb{R}$ .

(a) (i) Find an expression for  $f'(x)$ . [2]

(ii) Show that  $f'(x)g'(x) = \frac{4x}{(2-x)^3}$ . [2]

(iii) Show that  $f(x)g'(x) + g(x)f'(x) = \frac{4x}{(2-x)^3}$ . [4]

In parts (b) and (c), consider two non-constant functions,  $f(x)$  and  $g(x)$ , where  $f(x) > 0$  and  $g(x) \neq g'(x)$ .

(b) By rearranging the equation  $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$ , show that  $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ . [2]

**(This question continues on the following page)**

**(Question 1 continued)**

- (c) Hence, by integrating both sides of  $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ , show that  $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$ , where  $A$  is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions,  $f(x)$  and  $g(x)$ , which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } (f(x)g(x))' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with  $A = 1$ .

- (d) Consider  $g(x) = xe^x$ .

Find  $f(x)$  such that  $f(x)$  and  $g(x)$  satisfy the above two equations. [5]

- (e) Consider  $g(x) = \sin x + \cos x$ .

Find  $f(x)$  such that  $f(x)$  and  $g(x)$  satisfy the above two equations over the domain  $0 < x < \pi$ .

Give your answer in the form  $f(x) = \sqrt{e^x h(x)}$ , where  $h(x)$  is a function to be determined. [7]

2. [Maximum mark: 31]

**This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of  $x$ -intercepts.**

In parts (a) – (f), consider quadratic functions,  $f(x) = ax^2 + bx + c$ , whose coefficients,  $a$ ,  $b$  and  $c$ , are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a 2, 3 and 5 in turn generates the quadratic function  $f(x) = 2x^2 + 3x + 5$ .

(a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

(b) The set of coefficients,  $a = 1$ ,  $b = 4$  and  $c = 4$ , is randomly generated to form the quadratic function  $f(x) = x^2 + 4x + 4$ .

Verify that this graph of  $f$  has only one  $x$ -intercept. [2]

(c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one  $x$ -intercept is  $\frac{5}{216}$ . [6]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct**  $x$ -intercepts.

(d) By considering the discriminant, determine the set of possible values of  $ac$ . [3]

(e) (i) For the case where  $ac = 1$ , show that there are four quadratic functions whose corresponding graphs have two distinct  $x$ -intercepts. [1]

(ii) For the case where  $ac = 2$ , show that there are eight quadratic functions whose corresponding graphs have two distinct  $x$ -intercepts. [2]

Let  $p$  be the probability of the graph of such a randomly generated quadratic function having two distinct  $x$ -intercepts.

(f) Using the approach started in part (e), or otherwise, find the value of  $p$ . [6]

**(This question continues on the following page)**

**(Question 2 continued)**

In parts (g) and (h), consider a randomly generated quadratic function,  $f(x) = x^2 + 2Zx + 1$ , where the continuous random variable  $Z \sim N(0, 1)$ .

- (g) Find the probability that the graph of  $f$  has two  $x$ -intercepts. [3]

The continuous random variables,  $X_1$  and  $X_2$ , represent the  $x$ -intercepts of the graph of  $f$  where  $X_1 = -Z - \sqrt{Z^2 - 1}$  and  $X_2 = -Z + \sqrt{Z^2 - 1}$ .

- (h) Given that the graph of  $f$  has two  $x$ -intercepts,  $X_1$  and  $X_2$ , find the probability that both  $X_1$  and  $X_2$  are greater than 0.5. [7]
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