8824 – 9701M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2024

Mathematics: analysis and approaches

Higher level

Paper 2

30 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) (i) $f(0) = -11$ ***A1***

(ii) $-1.80650...$

$f(20) = -1.81 \left(= 11\sqrt{20} - 51 = 22\sqrt{5} - 51\right)$ ***A1***

[2 marks]

(b) attempt to find at least one root ***(M1)***

$x = 1.72622...$ and $x = 17.5237...$

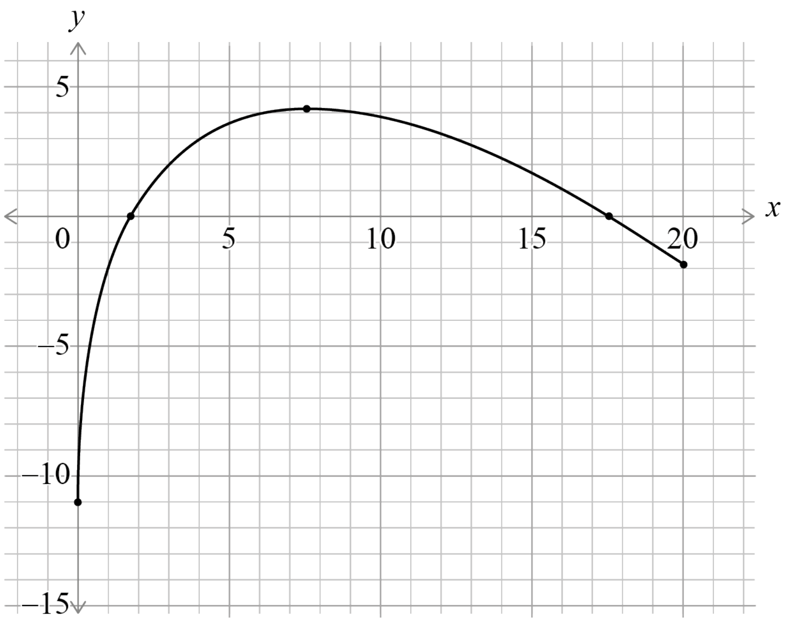
$x = 1.73$ and $x = 17.5$ ***A1***

[2 marks]

continued…

Question 1 continued.

(c)



A1A1A1

Note: Follow through from their part (a).

Award ***A1*** for endpoints at approximately $(0,-11)$ and $(20,-1.8)$. Allow for y-intercept $-12 < y < -10$ and for the right endpoint in the interval $x = 20, -2 < y < -1$.

The following two ***A*** marks may only be awarded if the approximate shape is correct.

Award ***A1*** for x-intercepts at approximately $x = 1.7$ and $x = 17.5$ and award ***A1*** for maximum at approximately $(7.6, 4.1)$. Allow for x-intercepts in the intervals $1 < x < 2$, $17 < x < 18$, and maximum in the intervals $6.5 < x < 8.5$, $3.5 < y < 5$.

[3 marks]

Total [7 marks]

2. **EITHER**

attempt to form a product of binomial coefficient, a power of $2x$ and a power of -5 seen ***(M1)***

${}^{9}C_{3}(2x)^{6}(-5)^{3}$ OR ${}^{9}C_{6}(2x)^{6}(-5)^{3}$ OR $84\times(2x)^{6}(-5)^{3}$ ***(A1)(A1)***

Note: Award ***A1*** for ${}^{9}C_{6}$ or ${}^{9}C_{3}$ or 84, ***A1*** for $(2x)^{6}(-5)^{3}$.

OR

attempt to use the general term ***(M1)***

${}^{9}C_{r}(2x)^{9-r}(-5)^{r}$ and $r=3$ ***(A1)(A1)***

THEN

-672000 (exact) ***A1***

Note: Award ***A0*** for a final answer of $-672000x^{6}$.

[4 marks]

3. (a) recognition of sum of probabilities equals 1 ***(M1)***

$$\frac{3k}{20}+\frac{5k}{20}+\frac{8k}{20}+\frac{11k}{20}=1$$

$$k=0.740740$$

$$k=0.741\left(=\frac{20}{27}\right)$$ ***A1***

[2 marks]

(b) correct probabilities: $\frac{3}{27},\frac{5}{27},\frac{8}{27},\frac{11}{27}$ OR $0.111,\ 0.185,\ 0.296,\ 0.407$ ***(A1)***

substitution of their probabilities into formula for expected value ***(M1)***

$$3\times\frac{3}{27}+5\times\frac{5}{27}+8\times\frac{8}{27}+11\times\frac{11}{27}\ \text{ OR }\ \frac{219k}{20}$$

$$=8.11111...$$

$$\mathrm{E}(X)=8.11\left(=\frac{219}{27}=\frac{73}{9}\right)$$ (same 3sf from previous 3sf answer) ***A1***

[3 marks]

Total [5 marks]

4. (a) **METHOD 1**

attempt to use right triangle trigonometry ***(M1)***

$\tan \mathrm{B\hat{A}E} = \frac{12}{7}$ OR $\tan\left(90° - \mathrm{B\hat{A}E}\right) = \frac{7}{12}$ ***(A1)***

$59.7435...$

$\mathrm{B\hat{A}E} = 59.7°$ ***A1***

Note*:* Award ***(M1)(A1)A0*** for the equivalent radian value of 1.04.

METHOD 2

attempt to find $\mathrm{B\hat{A}E}$ using sine rule OR cosine rule (***M1)***

$\frac{\sin \mathrm{B\hat{A}E}}{12} = \frac{\sin 90}{\sqrt{12^2 + 7^2}}$ OR $12^2 = 7^2 + 193 - 2 \times 7 \times \sqrt{12^2 + 7^2} \times \cos \mathrm{B\hat{A}E}$ ***(A1)***

$\mathrm{B\hat{A}E} = 59.7435...$

$\mathrm{B\hat{A}E} = 59.7°$ ***A1***

Note*:* Award ***(M1)(A1)A0*** for the equivalent radian value of 1.04.

[3 marks]

(b) (i) **METHOD 1**

attempt to find DE using right angle trigonometry ***(M1)***

$\sin 59.7435...° = \frac{350}{\mathrm{DE}}$ OR equivalent ***(A1)***

$\mathrm{DE} = 405.196...$

$\mathrm{CE} = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

METHOD 2

Let $\mathrm{DE} = \mathrm{EF} = x$

attempt to find DE using their $\mathrm{D\hat{E}F}$ and the sine rule OR cosine rule (***M1)***

$\frac{700}{\sin(119.487...)} = \frac{\mathrm{DE}}{\sin(30.2564...)}$ OR $x^2 = 700^2 + x^2 - 2 \times 700 \times x \times \cos 30.2564...$ ***(A1)***

$\mathrm{DE} = 405.196...$

$\mathrm{CE} = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

continued…

Question 4 continued.

METHOD 3

Let G be the midpoint of DF

$\mathrm{EG} = \frac{7}{12} \times 350 \left(= \frac{1225}{6} = 204.166... \right)$ ***(A1)***

use of Pythagoras' with their EG to find DE ***(M1)***

$\mathrm{DE} = \sqrt{204.166...^2 + 350^2} \; (= 405.196...)$

$\mathrm{CE} = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

(ii) $\tan(59.7435...°) = \frac{30}{x}$ OR $\frac{12}{7} = \frac{30}{x}$ ***(A1)***

$x = 17.5$

$\mathrm{BA} = 455.196... + 17.5$

$= 472.696...$

$= 473$ (cm) ***A1***

[5 marks]

Total [8 marks]

5. **METHOD 1**

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

(discriminant =) $(-r)^2 - 4(4)(r-1)$ $\left(= r^2 - 16r + 16\right)$ (seen anywhere) ***(A1)***

$1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $4x^2 - rx + r - 1$ is less than zero ***(M1)***

$1.07 < r < 14.9$ $(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3})$ ***A1***

Note: Accept $1.08 \le r \le 14.9$.

METHOD 2

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when $x = \frac{r}{8} \Rightarrow (y =)\, 4\left(\frac{r}{8}\right)^2 - r\left(\frac{r}{8}\right) + r - 1$ (> 0) ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{4x^2 - 1}{x - 1}$ OR $x > 1 \Rightarrow r < \frac{4x^2 - 1}{x - 1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{4x^2 - 1}{x - 1}$ ***(M1)***

THEN

$(r >)\, 1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $(r <)\, 14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$1.07 < r < 14.9$ $\left(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3}\right)$ ***A1***

Note: Accept $1.08 \le r \le 14.9$.

[5 marks]

6. (a) $\mathrm{E}(X)=\int_0^2 \frac{x}{5}\,\mathrm{d}x+\int_2^8\left(-\frac{x^2}{30}+\frac{4x}{15}\right)\mathrm{d}x$ ***(A1)(A1)***

Note: Award ***(A1)*** $\int_0^2 \frac{x}{5}\,(\mathrm{d}x)$ and ***(A1)*** for $\int_2^8\left(-\frac{x^2}{30}+\frac{4x}{15}\right)(\mathrm{d}x)$

$=\frac{2}{5}+\frac{12}{5}$

$=\frac{14}{5}(=2.8)$ ***A1***

[3 marks]

(b) attempt to use the expectation formula $\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$ ***(M1)***

$\mathrm{E}(c-2X)=c-2\mathrm{E}(X)(=0)$

$c=2\mathrm{E}(X)$

$=\frac{28}{5}(=5.6)$ ***A1***

[2 marks]

(c) recognition that median m lies between 2 and 8 e.g. using a diagram or integral ***(M1)***

$\int_0^2 \frac{1}{5}\mathrm{d}x+\int_2^m\left(-\frac{x}{30}+\frac{4}{15}\right)\mathrm{d}x=\frac{1}{2}$ OR $\int_m^8\left(-\frac{x}{30}+\frac{4}{15}\right)\mathrm{d}x=\frac{1}{2}$ OR $\int_2^m\left(-\frac{x}{30}+\frac{4}{15}\right)\mathrm{d}x=\frac{1}{10}$ ***(A1)***

$m=2.52277...$

$m=2.52$ ***A1***

[3 marks]

Total [8 marks]

7. (a) total ways $= 3!\,{}^{12}C_3 \left(= {}^{12}P_3 = 1320\right)$ OR total ways together $= 3! \times 10 \left(= 60\right)$ ***(A1)***

attempt to consider the total ways of sitting – total ways of sitting together ***(M1)***

$3!\,{}^{12}C_3 - 3! \times 10$

$= 1260$ ***A1***

[3 marks]

(b) **METHOD 1**

attempt to multiply ways of seating AVP by ways of sitting additional people ***(M1)***

AVP can sit in $3! \times 10 \left(= 60\right)$ ways (may be seen in part (a))

other 3 then have $9 \times 8 \times 7 \left(= {}^{9}P_3\right)$ ways to sit ***(A1)***

total ways $= 3! \times 10 \times 9 \times 8 \times 7$

$= 30240$ ***A1***

Note: Award ***(M1)(A0)A0*** for $3! \times 10 \times {}^{9}C_3 = 5040$.

METHOD 2

attempt to consider 'AVP' as one item, so 4 'items' in total ***(M1)***

${}^{10}C_4 \times 3! \times 4! \left(= {}^{10}P_4 \times 3! \right)$ ***(A1)***

$= 30240$ ***A1***

[3 marks]

Total [6 marks]

8. (a) **METHOD 1**

suppose $w = x + \mathrm{i}y$

$ww^* = (x+\mathrm{i}y)(x-\mathrm{i}y)$ ***A1***

$= x^2 + y^2$ ***A1***

$= |w|^2$ ***AG***

METHOD 2

suppose $w = r\mathrm{e}^{\mathrm{i}\theta}$

$ww^* = \left(r\mathrm{e}^{\mathrm{i}\theta}\right)\left(r\mathrm{e}^{-\mathrm{i}\theta}\right)$ ***A1***

$= r^2$ ***A1***

$= |w|^2$ ***AG***

METHOD 3

suppose $w = r(\cos\theta + \mathrm{i}\sin\theta)$

$ww^* = \left(r(\cos\theta + \mathrm{i}\sin\theta)\right)\left(r(\cos\theta - \mathrm{i}\sin\theta)\right)$ ***A1***

$= r^2\left(\cos^2\theta + \sin^2\theta\right)\left(= r^2\right)$ ***A1***

$= |w|^2$ ***AG***

[2 marks]

(b) **EITHER**

multiplying first equation by w OR multiplying LHS and RHS of both equations ***M1***

$5w^*w = (1-2\mathrm{i})z^2w$ OR $5w^*zw = (1-2\mathrm{i})z^2(10+10\mathrm{i})$

$5(2\sqrt{5})^2 = (1-2\mathrm{i})(10+10\mathrm{i})z\left(\Rightarrow 100 = (30-10\mathrm{i})z\right)$ ***A1***

OR

attempt to eliminate w and w^* using $ww^* = 20$ ***M1***

$$\left(\frac{(1-2\mathrm{i})}{5}z^2\right)\left(\frac{10+10\mathrm{i}}{z}\right) = 20$$

$$\frac{(1-2\mathrm{i})(10+10\mathrm{i})}{5}z = 20$$ ***A1***

continued…

Question 8 continued.

THEN

$$z=\frac{10}{(1-2\mathrm{i})(1+\mathrm{i})}\left(=\frac{10}{3-\mathrm{i}}\right)$$ ***(A1)***

$$z=3+\mathrm{i}$$ ***A1***

$$(a=3,\ b=1)$$

[4 marks]

Total [6 marks]

9. (a) recognition that $|\log_2 c|<1$ ***(M1)***

$0.5<c<2,\quad (c\neq 1)$ ***A1A1***

Note: Award ***A1*** for endpoints and ***A1*** for strict inequalities.

[3 marks]

(b) attempt to find $S_\infty=\dfrac{u_1}{1-r}$ ***(M1)***

$$=\frac{5}{1-\log_2(1.5)}\left(=\frac{5}{1-0.58496\ldots}=12.0471...\right)$$ ***(A1)***

attempt to solve their $|S_\infty-S_n|<0.1$ ***(M1)***

$$\left|\frac{5}{1-\log_2(1.5)}-\sum_{r=0}^{n-1}5\left(\log_2(1.5)\right)^r\right|<0.1 \text{ OR } \left|\frac{5}{1-\log_2(1.5)}-\frac{5\left(1-\left(\log_2(1.5)\right)^n\right)}{1-\log_2(1.5)}\right|<0.1$$

Note: Award ***(M1)*** for solving an equality. Condone absence of absolute value signs.

$n=8.93574...$

$n=9$ ***A1***

[4 marks]

Total [7 marks]

Section B

10. (a) (i) $a = 0.358$ (exact); $b = 30.5$ (exact answer is 30.52) ***A1A1***

Note: Award ***A1A0*** if the values of a and b are interchanged or not labeled.

(ii) a represents the (average) rate of increase (change) in population (0.358 millions of people per year). (or equivalent) ***R1***

[3 marks]

(b) It is unreliable because 2030 is outside the range of data (extrapolation). ***A1***

[1 mark]

(c) (i) attempt to find $B(100)$ ***(M1)***

$55.1633\ldots$

55.2 million OR $55{,}200{,}000$ ***A1***

(ii) The annual growth rate of the population is 0.5%. ***A1***

Note: Description must include some reference to annual rate.

[3 marks]

(d) $54.6094\ldots$

54.6 million OR $54{,}600{,}000$ ***A1***

[1 mark]

(e) consideration of the difference function $C(t) - B(t)$ or $B(t) - C(t)$ or $|C(t) - B(t)|$ ***(M1)***

evidence of finding the maximum (or minimum) of this function. ***(M1)***

$t = 58.6283\ldots$

2058 (accept 2059) ***A1***

[3 marks]

continued…

Question 10 continued.

(f) (i) $0.242876\ldots$
$B'(75) = 0.243$ ***A1***

(ii) $0.184941...$
$C'(75) = 0.185$ ***A1***

[2 marks]

(g) $B'(75) > C'(75)$ (or equivalent in words) ***A1***

the population in Benoit's model is increasing at a faster rate than in Cecilia's model (in 2075) (or equivalent) ***R1***

Note: Do not award ***A0R1***.

[2 marks]

Total [15 marks]

11. (a) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$-\frac{x}{2}+1=\lambda \Rightarrow x=2-2\lambda$, $y+4=\lambda \Rightarrow y=-4+\lambda$, $\frac{z}{3}=\lambda \Rightarrow z=3\lambda$ OR

$\frac{x-2}{-2}=\frac{y+4}{1}=\frac{z-0}{3}$

correct direction vector x, y, z or equivalent seen in vector form ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}2\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\3\end{pmatrix}$ ***A1***

Note: Award ***(M1)(A1)A0*** if "$\boldsymbol{r}=$" is omitted.

[3 marks]

(b) **METHOD 1**

L passes through $\mathrm{P}(2-2\lambda, -4+\lambda, 3\lambda)$ ***(A1)***

attempt to apply distance formula to find their $\left|\overrightarrow{\mathrm{OP}}\right|$ or their $\left|\overrightarrow{\mathrm{OP}}\right|^2$ ***(M1)***

$\left|\overrightarrow{\mathrm{OP}}\right|^2=(2-2\lambda)^2+(-4+\lambda)^2+(3\lambda)^2$ ***(A1)***

attempt to find their minimum value of $\left|\overrightarrow{\mathrm{OP}}\right|$ or $\left|\overrightarrow{\mathrm{OP}}\right|^2$ using GDC ***(M1)***

$\lambda=0.571428...\left(\Rightarrow\left|\overrightarrow{\mathrm{OP}}\right|=\sqrt{15.4285...}\text{ OR }\left|\overrightarrow{\mathrm{OP}}\right|^2=15.4285...\right)$

$3.92792...$

$3.93\ \left(=\frac{6\sqrt{21}}{7}\right)$ ***A1***

continued…

Question 11 continued.

METHOD 2

setting the scalar product of their line and their direction vector to zero ***(M1)***

$$\begin{pmatrix} 2-2\lambda \\ -4+\lambda \\ 3\lambda \end{pmatrix} . \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$-4+4\lambda-4+\lambda+9\lambda=0$$

$$-8+14\lambda=0$$

$$\lambda=\frac{4}{7}(=0.571428...)$$ ***(A1)***

attempt to substitute their value for λ into their $\boldsymbol{r}$ to find the position vector of the closest point and find $|\boldsymbol{r}|$ ***(M1)***

$$\boldsymbol{r}=\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}+\frac{4}{7}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}=\begin{pmatrix} \frac{6}{7} \\ -\frac{24}{7} \\ \frac{12}{7} \end{pmatrix}$$

$$|\boldsymbol{r}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(-\frac{24}{7}\right)^2+\left(\frac{12}{7}\right)^2}$$ ***(A1)***

$$=3.92792\ldots=\frac{6\sqrt{21}}{7}$$

$$=3.93\left(=\frac{6\sqrt{21}}{7}\right)$$ ***A1***

continued…

Question 11 continued.

METHOD 3

Let P be a point on L

attempt to find the cross product between $\overrightarrow{\text{OP}}$ and the direction, $\boldsymbol{b}$, of L ***(M1)`***

$$\begin{pmatrix} 2 \\ -4 \\ -0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \left(= (-12-0)\boldsymbol{i} + (0-6)\boldsymbol{j} + (2-8)\boldsymbol{k}\right)$$

$$= \begin{pmatrix} -12 \\ -6 \\ -6 \end{pmatrix} \left(= -6\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right)$$ ***(A1)***

attempt to find shortest distance using $\dfrac{\left|\overrightarrow{\text{OP}} \times \boldsymbol{b}\right|}{|\boldsymbol{b}|}$ ***(M1)***

$\dfrac{\sqrt{216}}{\sqrt{14}}$ ***(A1)***

$3.92792...$

$3.93 \left(= \dfrac{6\sqrt{21}}{7}\right)$ ***A1***

[5 marks]

continued…

Question 11 continued.

(c) **METHOD 1**

substitute their x, y, z into equation of plane $6x-3y+5z(=24)$ ***M1***

$6(2-2\lambda)-3(-4+\lambda)+5(3\lambda)$ ***(A1)***

$=12-12\lambda+12-3\lambda+15\lambda$ ***A1***

$=24$

so the line is contained in the plane ***AG***

Note: For FT from an incorrect part a), award ***M1A0A0.***

METHOD 2

consider the direction of the line **and** a point on the line ***M1***

$\begin{pmatrix}-2\\1\\3\end{pmatrix}.\begin{pmatrix}6\\-3\\5\end{pmatrix}=-12-3+15=0$ (so line is parallel to plane) ***A1***

$6(2)-3(-4)+5(0)=12+12=24$ (so line lies in the plane) ***A1***

so the line is contained in the plane ***AG***

Note: Both ***A*** marks are dependent on the ***M*** mark.
Note: For FT from an incorrect part a), award ***M1A0A0.***

[3 marks]

continued…

Question 11 continued.

(d) **METHOD 1**

recognition that the direction M is from the point $\begin{pmatrix}4\\1\\2\end{pmatrix}$ and a point on the z-axis $\begin{pmatrix}0\\0\\z\end{pmatrix}$ ***(M1)***

direction of M is $(\pm)\begin{pmatrix}4\\1\\2-z\end{pmatrix}$ ***(A1)***

the direction of the normal of Π is $\begin{pmatrix}6\\-3\\5\end{pmatrix}$ (seen anywhere) ***(A1)***

attempt to use the scalar product with their normal and their direction vector and equate to 0 ***(M1)***

$$\begin{pmatrix}4\\1\\2-z\end{pmatrix}.\begin{pmatrix}6\\-3\\5\end{pmatrix}=0\Rightarrow 24-3+10-5z=0$$

$z=\frac{31}{5}$ ***(A1)***

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}4\\1\\-\frac{21}{5}\end{pmatrix}$$

attempt to express equation in the form $\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}p\\q\\r\end{pmatrix}$ ***(M1)***

$\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-\frac{21}{5}\end{pmatrix}$ OR $\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}20\\5\\-21\end{pmatrix}$ ***A1***

continued…

Question 11 continued.

METHOD 2

let the equation of M be $M: \boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Note: Consideration of the z-axis intersection and consideration of direction may be done in either order and marks should be awarded independently.

recognition M intersects the z-axis where $x = y = 0$ ***(M1)***

$4 + \mu p = 0, 1 + \mu q = 0$

$\frac{-4}{p} = \frac{-1}{q} \Rightarrow p = 4q$ ***(A1)***

the direction of the normal of Π is $\begin{pmatrix} 6 \\ -3 \\ 5 \end{pmatrix}$ (seen anywhere) ***(A1)***

attempt to use the scalar product with their normal and their direction vector and equate to 0 ***(M1)***

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 6 \\ -3 \\ 5 \end{pmatrix} = 0 \Rightarrow \ 6p - 3q + 5r = 0$

$6(4q) - 3q + 5r = 0 \Rightarrow 21q = -5r$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$ (or any multiple of $\begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$) ***(A1)***

attempt to express equation in the form $\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ ***(M1)***

$\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -\frac{21}{5} \end{pmatrix}$ OR $\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$ ***A1***

[7 marks]

Total [18 marks]

12. (a) attempt to use quotient rule *M1*

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x+k)(4x+6)-(2x^2+6x-3)}{(x+k)^2}$$ *A1A1*

Note: Award ***A1*** for $=\frac{(x+k)(4x+6)}{(x+k)^2}$, ***A1*** for $\frac{-(2x^2+6x-3)}{(x+k)^2}$ OR $\frac{-2x^2-6x+3}{(x+k)^2}$

$$=\frac{4x^2+4kx+6x+6k-2x^2-6x+3}{(x+k)^2}$$ or equivalent leading to ***AG*** line *A1*

$$=\frac{2x^2+4kx+6k+3}{(x+k)^2}$$ *AG*

Note: Candidates may use product rule, give marks accordingly.

[4 marks]

(b) local min or max when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, when $2x^2+4kx+6k+3=0$ has real solutions.

attempt to find the discriminant of numerator (*M1*)

$(4k)^2-4\times2(6k+3)\left(=2k^2-6k-3<0\right)$ *(A1)*

attempt to find the critical values of their quadratic equation or inequality (*M1*)

$k=3.43649\ldots$ OR $k=-0.436491\ldots$ OR $k=\frac{3\pm\sqrt{15}}{2}$

$k>3.44\left(=\frac{3+\sqrt{15}}{2}\right)$ (since k is positive) *A1*

Note: Accept $k\geq3.44$.

[4 marks]

(c) $x=-2$ *A1*

[1 mark]

continued…

Question 12 continued.

(d) **METHOD 1**

$\frac{2x^2+6x-3}{x+2}=2x+...$ ***(A1)***

attempts division on $\frac{2x^2+6x-3}{x+2}$ ***M1***

$\frac{2x^2+6x-3}{x+2}=2x+\frac{2x-3}{x+2}$

$=2x+2+...$ ***(A1)***

asymptote is $y=2x+2$ ***A1***

METHOD 2

let asymptote be $y=ax+b$

$\frac{2x^2+6x-3}{x+2}=ax+b+\frac{c}{x+2}$ ***M1***

$2x^2+6x-3=(ax+b)(x+2)+c$

equates coefficients of x^2 and x: ***(M1)***

$a=2$ ***A1***

$6=2a+b$

$b=2$ ***A1***

$(y=2x+2)$

continued…

Question 12 continued.

METHOD 3

$\dfrac{2x^2+6x-3}{x+2}=2x+...$ ***A1***

$\dfrac{2x^2+6x-3}{x+2}-2x=\dfrac{2x-3}{x+2}$

attempt to find the limit as $x\to\infty$ ***M1***

$\lim\limits_{x\to\infty}\dfrac{2x-3}{x+2}=2$ ***A1***

$y=2x+2$ ***A1***

[4 marks]

continued…

Question 12 continued.

(e) **METHOD 1**

EITHER

for $k = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + 8x + 15}{(x+2)^2}$ ***A1***

attempt to write $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $p + \frac{q}{(x+2)^2}$ OR to write $\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ in the form $\frac{q}{(x+2)^2}$ ***M1***

OR

$y = 2x + 2 - \frac{7}{x+2}$ ***A1***

Note: Follow through from their part (d).

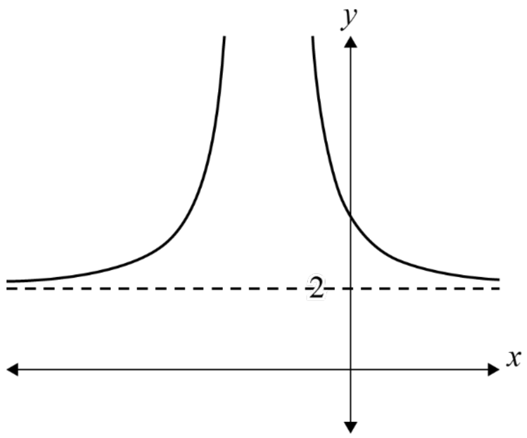
attempt to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = p + \frac{q}{(x+2)^2}$ ***M1***

THEN

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \frac{7}{(x+2)^2}$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} - 2 = \frac{7}{(x+2)^2}$ ***A1***

since $\frac{7}{(x+2)^2} > 0$ for $x \in \mathbb{R},\ x \neq -2$, $\frac{\mathrm{d}y}{\mathrm{d}x} > 2$. ***R1***

Note: Award a maximum of ***A1M1A0R0*** for a graphical approach which includes a horizontal asymptote at 2, showing that $\frac{\mathrm{d}y}{\mathrm{d}x} > 2$.



Question 12 continued.

METHOD 2

for $k = 2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x^2 + 8x + 15}{(x+2)^2}$ ***A1***

clear assumption that $\dfrac{2x^2 + 8x + 15}{(x+2)^2} \le 2$ ***M1***

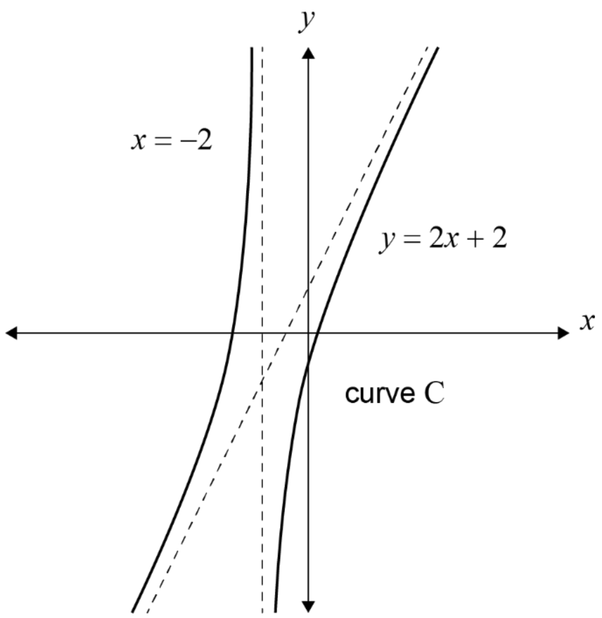
$\left(2x^2 + 8x + 15 \le 2(x+2)^2 \Rightarrow\right) 15 \le 8$ ***A1***

contradiction implies that $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 2$ ***R1***

Note: Award ***A1M0A0R0*** for solutions which work backwards from $\dfrac{2x^2 + 8x + 15}{(x+2)^2} > 2$ to arrive at $15 > 8$.

[4 marks]

(f)



curve C in approx correct place, with negative y-intercept and part of the curve in 4th quadrant ***A1***

Note: The following ***A1*** marks should be awarded independently **provided** there is an attempt to draw both branches of $y = f(x)$.

vertical asymptote in approx correct place ***A1***

oblique asymptote in approx correct place ***A1***

approx correct asymptotic behaviour of C relative to both asymptotes ***A1***

[4 marks]

Total [21 marks]