a) For any two vectors ***v*** and ***w*** prove Lagrange's Identity

$$|\mathbf{v}\times\mathbf{w}|^2+(\mathbf{v}\cdot\mathbf{w})^2=|\mathbf{v}|^2|\mathbf{w}|^2$$

b) **Hence**, find $\mathbf{v}\cdot\mathbf{w}$ if

$$|\mathbf{v}|=3$$

$$|\mathbf{w}|=4$$

$$\mathbf{v}\times\mathbf{w}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

a) From the formula booklet, we know

$$|\boldsymbol{v}\times\boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|sin\theta$$

$$\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}|cos\theta$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$|\boldsymbol{v}\times\boldsymbol{w}|^2 + (\boldsymbol{v}\cdot\boldsymbol{w})^2 = (|\boldsymbol{v}||\boldsymbol{w}|sin\theta)^2 + (|\boldsymbol{v}||\boldsymbol{w}|cos\theta)^2$$

$$= |\boldsymbol{v}|^2|\boldsymbol{w}|^2\sin^2\theta + |\boldsymbol{v}|^2|\boldsymbol{w}|^2\cos^2\theta$$

$$= |\boldsymbol{v}|^2|\boldsymbol{w}|^2(\sin^2\theta + \cos^2\theta)$$

$$= |\boldsymbol{v}|^2|\boldsymbol{w}|^2$$

b) $|\boldsymbol{v}\times\boldsymbol{w}|^2 + (\boldsymbol{v}\cdot\boldsymbol{w})^2 = |\boldsymbol{v}|^2|\boldsymbol{w}|^2$

$$(\boldsymbol{v}\cdot\boldsymbol{w})^2 = |\boldsymbol{v}|^2|\boldsymbol{w}|^2 - |\boldsymbol{v}\times\boldsymbol{w}|^2$$

$$\boldsymbol{v}\times\boldsymbol{w}=\begin{pmatrix}\mathbf{-1}\\\mathbf{2}\\\mathbf{3}\end{pmatrix}$$

$$\begin{aligned}|\boldsymbol{v}\times\boldsymbol{w}| &= \sqrt{(-1)^2+2^2+3^2}\\ &= \sqrt{1+4+9}\\ &= \sqrt{14}\\ |\boldsymbol{v}\times\boldsymbol{w}|^2 &= 14\end{aligned}$$

$$\begin{aligned}(\boldsymbol{v}\cdot\boldsymbol{w})^2 &= |\boldsymbol{v}|^2|\boldsymbol{w}|^2 - |\boldsymbol{v}\times\boldsymbol{w}|^2\\ &= 3^2\times4^2 - 14\\ &= 9\times16 - 14\\ &= 144 - 14\end{aligned}$$

$$= \quad 130$$

$$\boldsymbol{v} \cdot \boldsymbol{w} \quad = \quad \pm\sqrt{\mathbf{130}}$$