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Thinking out of the Box . . . Problem

It's a richer problem than we ever imagined

The "box problem" has been a standard optimization exercise in almost every calculus textbook since Leibniz and Newton invented calculus. With the capability of technology in the form of graphing calculators, this exercise has recently become standard fare earlier in the mathematics curriculum. We even find it in middle school curricula as a nice hands-on exercise in data analysis. With some variations in the numerical dimensions of the paper, the problem is similar to the following:

> Given a rectangular sheet of paper 8.5 inches × 11 inches, form a box by cutting congruent squares from each corner, folding up the sides, and taping them to form a box without a top. To make a box with maximum capacity, how large should the square cutouts from the corners of the original paper be? See **figure 1.**

Graphing calculators have allowed the box problem to become standard fare earlier in the curriculum

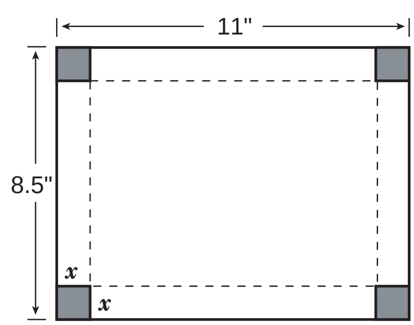

Fig. 1

In a middle school setting, groups of students are often given sheets of paper and asked to cut uniform squares from the corners, cutting different-sized squares for each sheet. They then fold up the sides to make a variety of boxes of different sizes and fill these boxes with something, such as popcorn, and measure or count the amount needed to fill the boxes. In this manner, students obtain a rough determination of the cutout size that results in a box with maximum capacity.

In a course prior to calculus, students might be asked to write the function of x that describes the volume of any box in which the length of the side of the square cutout is denoted by x. This function is as follows:

$$v(x) = x(8.5 - 2x)(11 - 2x)$$

Depending on the course and on the technology available, students can gather data from this function or graph it over the interval [0, 4.25] and thus determine the box of maximum volume, that is, the absolute maximum point of the data or of the graph over the given interval. A graph of this function, drawn with a TI-83 graphing calculator, is given in **figure 2.**

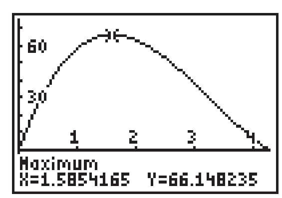

Fig. 2

In a beginning calculus course, students could use a symbolic manipulator or take the derivative of the volume function by hand, set it equal to 0, solve, and thus determine the x-value that gives the box of maximum volume:

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$$v(x) = x(8.5 - 2x)(11 - 2x)$$
$$v(x) = 93.5x - 39x^2 + 4x^3$$
$$v'(x) = 93.5 - 78x + 12x^2$$
$$0 = 93.5 - 78x + 12x^2$$
$$x \approx 1.585$$

or

$$x \approx 4.915$$

Discarding the solution that is not in the practical interval [0, 4.25] yields an approximate solution of x = 1.585 inches. We can verify this x-value yielding the absolute maximum by testing the endpoints where the volume is zero and the volume at x = 1.585 that is positive.

The previously described experiences usually comprise the total exposure that a student, or for that matter, a teacher, has with this problem that may also lead to interesting explorations for algebra and geometry students. This article reveals further questions that can be investigated from this simply stated problem. We assigned some of these questions to our AP calculus students as projects to complete outside of class. We have had fun developing some of the later questions on our own and plan to use them with students in the future.

Question 1

If we always start with a square sheet of paper, does a common relationship exist between the length of the side of this square and the length of the side of the smaller squares that are cut out from each corner?

We asked students to experiment with several square sheets of paper of different sizes, gather data, try to find a general relationship, and then prove that general relationship. In the interests of space, we give only the general solution for a square sheet of paper that measures a units by a units, as shown in **figure 3.**

The following work could also be done using a symbol manipulator.

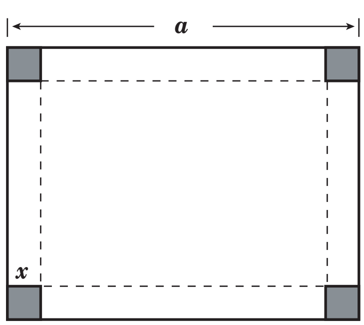

Fig. 3

Let $v(x) = x(a - 2x)^2$, where x is in the interval [0, a/2]. Then

$$v(x) = 4x^3 - 4ax^2 + a^2x,$$
$$v'(x) = 12x^2 - 8ax + a^2,$$
$$0 = 12x^2 - 8ax + a^2.$$

We see that it factors, so

$$0 = (6x - a)(2x - a)$$

and

$$x = \frac{a}{6}.$$

The other solution obviously yields a minimum volume.

The solution makes students realize that x is the variable for differentiation and that a, although a variable, is constant with respect to the differentiation, that is, it is one of those very useful—fixed but still variable—variables. This concept is also a precursor of multivariate calculus. In addition, we obtain a very simple general result, which says that to find the box of maximum volume starting with any square sheet of paper, we simply make the square cutouts at each corner with a side length that is one-sixth that of the side length of the original square.

The concept is a precursor of multivariate calculus

Question 2

If we start with a square but think dynamically of increasing one side of that square to form larger and larger rectangular sheets of paper while still keeping the adjacent side of fixed length, how does the side of the square cut out from the corners of this paper to form the box of maximum volume vary as this dynamic side becomes larger and larger?

For example, we consider sheets of paper of the following sizes: 6 inches × 6 inches, 6 inches ×

8 inches, 6 inches × 10 inches, 6 inches × 12 inches, and so on. We know that for the 6 inch × 6 inch square, we cut out 1 inch × 1 inch squares. Is the length of the cutout for the 6 inch × 8 inch square more than 1 inch, less than 1 inch, or still 1 inch? What happens to the cutout length as the variable side of the rectangle gets longer and longer? Does a limiting value exist? If so, what is it?

This question is much tougher but also much more rewarding for the persevering student

We ask our students to experiment by solving several concrete examples and obtaining a pattern, then generalizing, and finally maybe even proving their generalizations. This question is much tougher than the first one but also much more rewarding for the persevering student. In this article, we offer only a flavor of the total experience.

We assume that the original square sheet of paper is a units × a units and that the side denoted by b is the one that is increasing in size. We next want to find the value of x for any values of a and b that yield the box of maximum volume, as shown in **figure 4.**

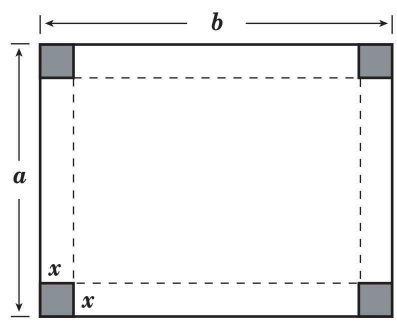

Fig. 4

Again, a symbolic manipulator can be used to do the following calculations.

Let $v(x) = x(a - 2x)(b - 2x)$, where x is in the interval $[0, a/2]$. Then

$$v(x) = 4x^3 - (2a + 2b)x^2 + abx,$$
$$v'(x) = 12x^2 - 4(a + b)x + ab,$$
$$0 = 12x^2 - 4(a + b)x + ab.$$

Solving by using the quadratic formula yields

$$x = \frac{(a + b) \pm \sqrt{a^2 - ab + b^2}}{6}.$$

For any value of $b \geq a$, the solution using the positive root is greater than or equal to $a/2$, so it is not the maximal solution that we desire. Hence, the maximal solution is given by

$$x = \frac{(a + b) - \sqrt{a^2 - ab + b^2}}{6}.$$

This equation solves any box problem, given the dimensions of the original sheet of paper, a and b. When $b = a$, it gives, as it should, the solution found in question 1.

To get an idea of the solution to the queries given in question 2, we asked our students to fix $a = 6$ and then consider x as a function of only b. The result is

$$x(b) = \frac{(6 + b) - \sqrt{36 - 6b + b^2}}{6}.$$

We next use a graphing calculator to make a graph of x as a function of b. On a TI-83 calculator, Y1 assumes the role of x, and X assumes the role of b. Therefore, the x-axis represents the length of the rectangle whose adjacent side is 6, and the y-axis represents the cutout size for the corner squares that yields the box with maximum volume:

$$y_1 = \frac{(6 + x) - \sqrt{36 - 6x + x^2}}{6}.$$

with a window of x: [0, 100] and y: [0, 3]. See **figure 5.**

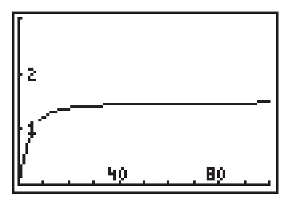

Fig. 5

We notice that as the side b increases beyond 6, the cutout size x, for the box with maximum volume also increases; but a limiting value, that is, a horizontal asymptote for the graph, does seem to exist. Using the table feature of the TI-83 in Ask mode (through TBLSET) and trying higher and higher values for b, we find that the limit for b seems to be 1.5 units. See **figure 6.**

We did ask students to try a couple of other fixed values for a so that they might see a general pattern. If students do so, they see that the cutout

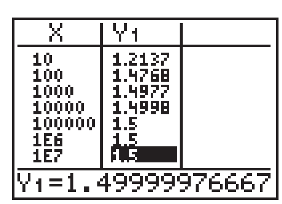

Fig. 6

value x always seems to approach $a/4$, where a is the dimension of the fixed side of the sheet of paper. A nice symbolic-manipulator exercise—for students using a TI-89, TI-92, or the like—is to have it actually evaluate the limit. In this situation, technology makes this limit accessible to a great number of students who would not be able to do the limit calculations by hand:

$$\lim_{b\to\infty} x = \lim_{b\to\infty} \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}.$$

Some more mathematically able students can evaluate this limit analytically by rationalizing the numerator as follows:

$$\lim_{b\to\infty} \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$$

$$= \lim_{b\to\infty} \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6} \times \frac{(a+b)+\sqrt{a^2-ab+b^2}}{(a+b)+\sqrt{a^2-ab+b^2}}$$

$$= \lim_{b\to\infty} \frac{3ab}{6(a+b+\sqrt{a^2-ab+b^2})}$$

$$= \lim_{b\to\infty} \frac{a}{2\left(\frac{a}{b}+1+\sqrt{\frac{a^2}{b^2}-\frac{a}{b}+1}\right)}$$

$$= \frac{a}{4}$$

We know that for any size $a \times b$ sheet of paper, where $a \le b$, the cutout size, x, for the length of the side of the square cut from each corner to yield the box of maximum volume always satisfies the inequality

$$\frac{a}{6} \le x < \frac{a}{4}.$$

The square sheet of paper uses the smallest cutout size; and the more elongated the paper is, the closer the cutout size should be to $a/4$.

Question 3

For any rectangular sheet of paper that measures $a \times b$, does a relationship exist between the lateral area and the area of the base for the box of maximal volume found by cutting congruent squares of side length x from the corners of the paper? If so, what is this relationship?

This question was one that we had not immediately considered. Only later did we begin to pursue the relationship between the lateral area and the area of the base of the box with maximal volume. This question becomes fundamental in the rest of our work. Again, students should experiment before seeking the formal result; however, this result is quite easy in its general form. We assume that the original sheet of paper is $a \times b$ with $a \le b$, and the cutout-square side length is again denoted by x. Where x is in the interval $[0, a/2]$,

$$v(x) = x(a-2x)(b-2x).$$

Rather than expanding the term on the right out to obtain a polynomial, we can take the derivative in this form using the product rule. One of us had done so initially and noted that the first term was the area of the base of the box and wondered whether the remaining term had any physical significance:

$$v'(x) = (a-2x)(b-2x) + x[(a-2x)(-2) + (b-2x)(-2)]$$

Rewriting this result in a slightly different form yields the following:

$$v'(x) = (a-2x)(b-2x) - x[2(a-2x) + 2(b-2x)]$$
$$0 = (a-2x)(b-2x) - x[2(a-2x) + 2(b-2x)]$$

We notice that $(a-2x)(b-2x)$ is the area of the base of our box and that $x[2(a-2x) + 2(b-2x)]$ is the lateral area of the box. Hence, when we have the box of maximal volume, the area of the base minus the lateral area equals 0. Therefore, the box of maximal volume is always the box that has the property that the lateral area is the same as the area of the base. This result gives us an easy way to verify whether any open box previously constructed from a rectangular sheet of paper is indeed a box with maximal volume. We simply measure the length, width, and height of the box and then calculate the base area and the lateral area. If the two results are equal, the box is the box with maximal volume; otherwise, it is not.

If we had relied only on a symbolic manipulator, we might not have been able to see this relationship. A symbolic manipulator gives only the symbolic form that has been programmed into it. A different form often gives one better insight into a generalization. In this situation, writing the equation in our special symbolic form enabled us to clearly see the relationship.

So far, we have just been using rectangular sheets of paper and have been cutting squares out at each corner to form a box. No reason dictates that the piece of paper must be rectangular.

Technology makes this limit accessible to a great number of students

Question 4

a) If the piece of paper that we start with is an equilateral triangle, how do we cut out the corners so that we can then fold up the sides and have a box that has an equilateral triangle for a base?

b) Once we have solved part (*a*), what is the relationship between the side of the original equilateral triangle and the height, x, of the lateral sides of the box formed in part (*a*) that gives the box of maximum volume?

We no longer cut squares with a side length of x out of the corners. Instead, we draw in the angle bisectors of the three angles and mark off the same distance on each one. We then connect the three endpoints of these angle-bisector segments. We can see the base of our solid in **figure 7.** Finally, we draw in the six perpendicular segments from these points to the original three sides of the paper. The length of these perpendicular segments is denoted by x. Hence, x becomes the height of our box. Thus, for the equilateral triangle, we cut out congruent kites from each corner. Each kite has two opposite right angles and the 60-degree angle from the original equilateral triangle paper.

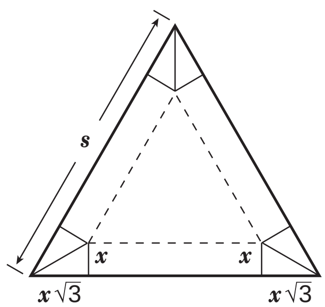

Fig. 7

To answer part (b) of question 4, we need to write a formula for the volume in terms of the original side, s, of the equilateral triangular paper and x. We can immediately write it in general, but students should do a few concrete examples first. We use the fact that the area of an equilateral triangle is given by

$$A = \frac{(\text{side})^2 \sqrt{3}}{4}.$$

Compare the height with the apothem of the regular polygonal sheet of paper

Then

$$v(x) = x(s - 2x\sqrt{3})^2 \frac{\sqrt{3}}{4},$$

where x is in the interval

$$\left[0, \frac{s\sqrt{3}}{6}\right].$$

We next take the derivative, set it equal to 0, and solve for x. Using a symbolic manipulator yields $x = s\sqrt{3}/18$ or $x = s\sqrt{3}/6$. Because the latter result obviously gives a minimum volume, our maximum occurs when $x = s\sqrt{3}/18$.

This result is not as satisfying as we had hoped. So far, we have worked with two regular polygons, the square and the equilateral triangle. For the square, the result was $x = s/6$. We were hoping for some simple relationship that would give us the result quickly for all regular polygons. The work became quite tedious, and the result did not seem to follow any simple pattern. We also tried to obtain the result for a regular hexagonal sheet of paper and the general regular n-gon sheet of paper. We used the formula $A = (1/2)ap$ for the area of the base of the box when writing the volume function. During these calculations, we finally arrived at a better way to look at our results. We decided to compare the height of the box of maximum volume with the apothem of the regular polygonal sheet of paper, not with the side, as we had been doing. Because the apothem is half the side of the square, the maximum box occurs when the cutout size is one-third of the apothem, or

$$\begin{aligned} x &= \frac{s}{6} \\ &= \frac{2a}{6} \\ &= \frac{a}{3}, \end{aligned}$$

where a is the apothem of the square and s is the side. We next look at the equilateral triangle, as shown in **figure 8.**

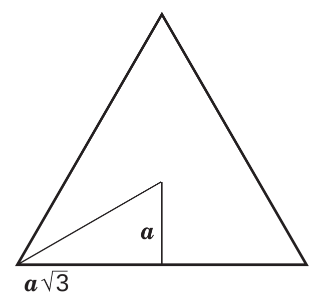

Fig. 8

From the diagram, we see that $s = 2a\sqrt{3}$, so that

$$\begin{aligned} x &= \frac{s\sqrt{3}}{18} \\ &= \frac{2a\sqrt{3}\sqrt{3}}{18} \\ &= \frac{a}{3}. \end{aligned}$$

For both the square and the equilateral triangle, the height of the box of maximum volume is one-third the length of the apothem of the original sheet of paper.

Question 5

For any regular n-gon sheet of paper, if congruent kites are cut from the corners and then the sides are folded up to form a box with a similar n-gon for a base, what is the relationship between the height of the box and the apothem of the original sheet of paper for the box of maximum volume?

We have an idea that the answer might be $x = a/3$, where x is the height of the box and a is the apothem of the original n-gon sheet of paper. To prove this result, perhaps we should write the area of the n-gon as a function of the apothem, a, of the original sheet of paper.

For the square.

$$\begin{aligned} v(x) &= x(2a - 2x)^2 \\ &= 4x(a - x)^2. \end{aligned}$$

We have already shown that the maximum occurs when $x = a/3$. Thus, if we take the derivative of $v(x)$ and set it equal to 0, our result will be $x = a/3$.

For the equilateral triangle. We know that $s = 2a\sqrt{3}$, so that

$$\begin{aligned} v(x) &= x(2a\sqrt{3} - 2x\sqrt{3})^2 \frac{\sqrt{3}}{4} \\ &= 3\sqrt{3}x(a - x)^2. \end{aligned}$$

We notice that this formula differs only by a constant factor from that of the volume of the square, so the derivative has the same roots. Hence, we again see that $x = a/3$ is the correct solution.

The general regular n-gon. From **figure 9** and using the fact that the area of any regular n-gon is given by $A = (1/2)ap$, where a is the apothem and p is the perimeter, we know that the volume of the box is

$$v(x) = \frac{1}{2}(a - x)n\left(2a \tan\left(\frac{\pi}{n}\right) - 2x \tan\left(\frac{\pi}{n}\right)\right)x;$$

$$v(x) = n \tan\left(\frac{\pi}{n}\right)x(a - x)^2.$$

We notice that this result is just a constant times the formula for the volume of the square; hence, again the box with maximum volume occurs when x, the height of the box, is chosen such that $x = a/3$, where a is the apothem of the original sheet of paper.

When we have the general regular n-gon volume formula, students can go back and try values of $n = 3$ and $n = 4$ and verify that these values are the exact constant factors that we determined earlier when we did these problems separately.

Question 6

What is the relationship between the lateral area and the area of the base of the box of maximum volume constructed from a regular n-gon sheet of paper?

This relationship is a relatively easy one to determine from the fact that the maximum volume box has $x = a/3$. Then the area of the base is

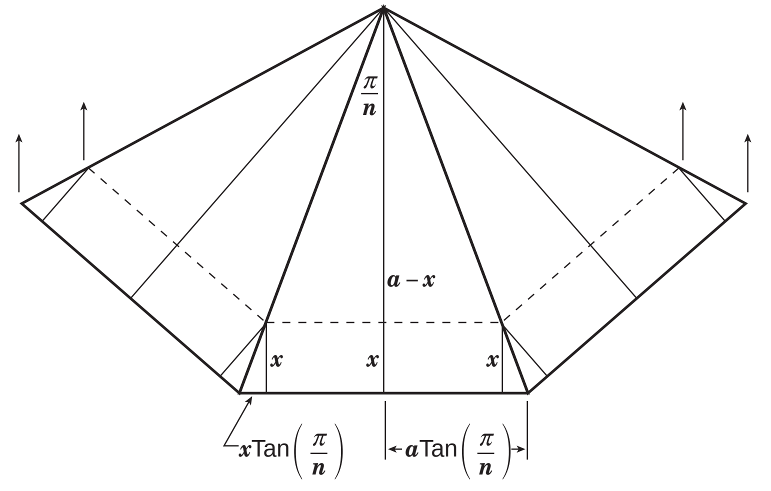

Fig. 9
A section of a regular n-gon

$$\begin{aligned} \frac{1}{2}(a - x)p &= \frac{1}{2}\left(a - \frac{a}{3}\right)p \\ &= \frac{1}{2}\left(\frac{2a}{3}\right)p \\ &= \frac{a}{3}p \\ &= xp, \end{aligned}$$

which is the lateral area. Hence, the box with maximum volume made from any regular n-gon–shaped sheet of paper has lateral area equal to the area of its base.

The formula differs only by a constant factor from that of the square, so the derivative has the same roots

SUMMARY

This classic problem has much more to offer than what appears in most textbooks. Along the way in our questioning, we used a great deal of high school mathematics, and we have discovered interesting geometric and algebraic relationships. Finally, we hope that when readers see a classic problem, they will think beyond that problem and try to find interesting mathematical generalizations lurking in the background.

FOR DISCUSSION WITH STUDENTS AND COLLEAGUES

We pose further questions that are related to the content of this article. We have explored answers to the first three of these questions and would be interested in seeing whether readers agree with our results and seeing how they obtained their results.

Extension 1

For any given sheet of paper that measures $a \times b$, where $a \le b$, if we always make the length, x, of the side of the cutout square, so that $x = a/5$, how far will we be from the box with maximum volume? From a practical standpoint, for all typical boxes that would normally be manufactured, we are really asking whether we could tell the production staff to always cut the squares at the corner of length $x = a/5$, where a is the shortest dimension of the original sheet of paper, and not be too far from the box with maximum volume.

Extension 2

For the regular n-gon situation, what happens in the limit as the number of sides of the regular n-gon approaches infinity? What relationships do we obtain for a cylinder?

Extension 3

We have seen that with regular n-gon paper and with rectangular paper, the box with maximal volume occurs when the lateral area of the box and the area of the base of the box are equal. Do we continue to obtain this result if we are given any convex polygon as the original sheet of paper? The reader should either furnish a proof or a counterexample.

Extension 4

What other extensions of this problem should students consider?

Extension 5

What are appropriate uses of computer algebra systems and other technology in these extensions? What can students learn about appropriate technology use from these box problems?

Extension 6

The teacher can choose another classic problem from the mathematics curriculum. What extension questions could be used with that problem? How might students respond to these questions? What mathematics would they learn or use in the solutions?

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